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BOŠKOVIĆ'S THEORY OF THE TRANSFORMATIONS OF GEOMETRIC LOCI: PROGRAM, AXIOMATICS, SOURCES

While preparing the third volume of his mathematical textbook *Elementorum universae matheseos,* Ruđer Bošković was systematically expounding the elements of conic sections. Here, among other things, he was analysing the transformations of conic sections one into another and, in particular, into a circle, as well as the degenerations of conic sections into straight lines and into a point. As a result, Bošković tackled the problem of the transformations of geometric loci in all its generality in his treatise *De transformatione locorum geometricorum,* and published it together with *Sectionum conicarum elementa* in the same third volume, in January of 1754.[1] Among Bošković's contemporaries, Bernard Zamagna was the one who surely realized the worth of this treatise, probably because he was a student of Bošković at the Collegium Romanum. In his commemorative oration held in the Cathedral on the 21st May 1787 by the decision of the Senate of the Republic of Dubrovnik, he began to evaluate the period of Bošković's Roman professorship just with the account of the third volume of *Elementorum universae matheseos.* In a brilliant Latin of Bernard Zamagna the estimation reads: "hinc illae novae elegantioresque coni sectionum explanationes; hinc illi novi curvarum linearum usus multiplicesque evolutiones".[2]

Bošković's treatise *De transformatione locorum geometricorum* has not yet been systematically researched. Željko Marković reviewed it in a few lines in perspective of Bošković's preface to the third volume of *Elementorum universae matheseos,*[3] while

[1] Ruđer Bošković to his brother Božo Bošković (Dubrovnik), 22 January 1754, Institute for the History of Science of the Yugoslav Academy of Sciences and Arts *(Zavod za povijest znanosti JAZU)* in Zagreb, Collection of Bošković's correspondence, Branimir Truhelka's transcription T-25, VIII, 43: "In questi giorni però ho faticato bestialmente. Al fine delle mie Sezioni Coniche mi è convenuto aggiugere una dissertazione sulle transformazioni de luoghi geometrici, e sull'infinito, la quale mi è tanto cresciuta sotto la penna, che è piu di un terzo del tomo. Lo stenderlo, far le figure, riveder la stampa, sono fatiche orribili, e tutto è affolato insieme. Spero, che uscirà la Settimana, che viene, ...". Cfr. Željko Marković, *Ruđe Bošković,* Vol. 1 (Zagreb: JAZU 1968), pp. 286-287; Ivica Martinović, "Pretpostavke za razumijevanje geneze Boškovićevih ideja o neprekinutosti i beskonačnosti: kronologija radova, povijesna samosvijest, tematske odrednice (Preliminaries for the genesis of Ruđer Bošković's ideas on continuity and infinity: chronology of writings, historical self-consciousness, thematic determinants," *Vrela i prinosi* 16 (1986), pp. 3-22, on pp. 9-10.

[2] Bernardus Zamagna, *Oratio in funere Rogerii Josephi Boscovichii* (Rhacusii, 1787), p. VI.

[3] Željko Marković, *Ruđe Bošković,* Vol. 1 (cit. n. 1), pp. 284-286.

Arnošt Kolman dealt with a part of Bošković's explanation of the eleventh canon "from the philosophical point of view", i.e. "not going now into the analysis of entirely geometrical questions".[4] Bošković's treatise on the geometric transformations was discussed indirectly by two scholars. Juraj Majcen "presented the elaboration" of Bošković's work *Sectionum conicarum elementa*.[5] His "elaboration", accompanied with a lenghty historical introduction, the evaluation of Bošković's contribution to the theory of conic sections, valuable notes, the thematic survey of the work, and the name index, touched on all those results that were used by Bošković later in the explanation of geometric transformations. Ernest Stipanić also explored Bošković's transformations of conic sections in his scientific and historical commentary on Bošković's treatise *De continuitatis lege* (1754), naturally in accordance with the choice Bošković himself made, from the extensive material on transformations of geometric loci in nn. 53-88, carrying out the ample induction in favour of the law of continuity.[6] The results of my research work are to be added here. Researching the beginnings of Bošković's mathematical work, I established that the assertion of the straight line as the circumference of the infinite circle, put forward for the first time in the treatise *De maris aestu* (1747), became the tenth canon of Bošković's theory of transformations just in his treatise *De transformatione locorum geometricorum*.[7] Further, I proved how Bošković's approach to Newton's method of prime and ultimate ratios influnced the formulation and commentary of the eleventh canon of Bošković's geometric theory.[8] Here, I want to study the program, structure, and sources of Bošković's investigations in the treatise *De transformatione locorum geometricorum*. The study of a kind is a necessary condition for a detailed analysis and estimation of Bošković's theory of the transformations of geometric loci.

The program of Research into Continuity and Infinity in Geometry

The subject of Bošković's treatise are the transformations of geometric loci. The notions of the geometric locus and its transformation, although the key notions of the

[4] A. Kolman, "Rudžer Boškovič i problema beskonečnosti," in *Actes du Symposium international R. J. Bošković 1961* (Beograd: Conseil des Academies RFPY, 1962), pp. 89-96, on pp. 92-95; on character of Kolman's approach see p. 92.

[5] Juraj Majcen, "Matematički rad Boškovićev II. dio. Sectionum conicarum elementa (Boškovićeva teorija krivuljâ 2. reda)," *Rad JAZU* 225 (1921), pp. 1-231. From the point of view of the history of science, Majcen's method is at least dubious. This edition is neither a critical edition, nor a translation, nor the complete text of *Sectionum conicarum elementa*, but, as the author himself says on pp. 5-6, "elaboration" (*"obrada"*), consequently, Majcen's distinctive adaptation of Bošković's text. Majcen estimated it on the basis of the comparison with the earlier and later written theories of conic sections and lectured on it in the seminar "Bošković's theory of curves of the second order (Boškovićeva teorija krivulja 2. stepena)" during the summer semester of the academic year 1920/1921. See Vilim Niče, "Juraj Majcen," *Rad JAZU* 325 (1963), pp. 49-106, on p. 93.

[6] Ernest Stipanić, "Naučni i istorijski komentar (A scientific and historical commentary on *De continuitatis lege*)," in Ruđer Bošković, *O zakonu kontinuiteta i njegovim posledicama u odnosu na osnovne elemente materije i njihove sile* (Beograd: Matematički institut, 1975), pp. 93-158, on pp. 121-134.

[7] Ivica Martinović, "Boškovićev prijepor o jednostavnosti pravca iz god. 1747.: izrečeni i prešućeni argumenti (Bošković's controversy on the simplicity of straight line from the year 1747: said and unsaid arguments)," *Vrela i prinosi* 16 (1986), pp. 167-179, on pp. 172-174.

[8] Ivica Martinović, "Završni sholij Newtonove metode prvih i posljednjih omjera i jedanaesti kanon Boškovićeve teorije geometrijskih transformacija (The last scholium of Newton's method of prime and ultimate ratios and the eleventh canon of Bošković's theory of geometrical transformations)," in Ljubisav Novaković et al. (eds), *Newtonova philosophia naturalis: nastanak i prevazilaženje* (Kragujevac: Institut za fiziku, 1987), pp. 157-171.

treatise, are not defined in the very text of the treatise, but are supposed to be known. However, Bošković seems to have felt that he had deviated from his customary approach under a flood of ideas. Therefore, he inserted his additional reflections on the fundamental notions of *De transformatione locorum geometricorum* in the texts he wrote during the same year.

In the treatise *De continuitatis lege* Bošković conceived the geometric loci as "infinite kinds of continious lines".[9] The nature of these curves is simple. General properties of curve, originating from its nature, are also simple and hold for every, even the smallest part of a curve. A part of a continuous curve, as a part of a geometric locus, is also mentioned at the beginning of Bošković's treatise of transformation.[10] It is pointed out where the nature of a curve originates from: from its definition. The form of the definition, as far as it can be understood, is not an analytic expression, but a form typical of the geometric synthetic method, e.g. the form of the definition at the beginning of *Sectionum conicarum elementa* where an ellipsis, a parabola, and a hyperbola are introduced by means of a constant defining ratio.[11] The similar definition occurs in *De transformatione locorum geometricorum,* where the ratios between the abscissa and the ordinate in any rational relation are introduced, which applies in particular to the family of parabole and hyperbolae of a higher order *(Parabolarum, ac Hyperbolarum familia)*.[12] Although this other form of the definition was so close to the analytic formula, Bošković remarked that there is no raising over the third power in geometry, while the raising to any power is possible in arithmetic consideration.

Bošković explained the notion of transformation in the preface to the third volume of *Elementorum universae matheseos,* immediately after the commpletion of the treatise. From the sense and formulation of this explanation it can be extracted how Bošković comprehended the transformation:

> Let any problem be generally solved by the geometric method, i.e. by the construction. If the disposition of the given quantities is somewhat changed, then mostly the construction is bound to be changed, too, sometimes considerably.[13]

On the same occasion Bošković sets forth a typology of changes that arise in geometrical constructions due to transformations:

(1) the transition from a sum to a difference and vice versa;
(2) the change of the direction of straight lines and angles;

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[9] Rogerius Josephus Boscovich, *De continuitatis lege et ejus consectariis pertinentibus ad prima materiae elementa eorumque vires* (Romae: Salomoni, 1754), n. 53, p. 23: "In primis in Geometria sunt infinita linearum continuarum genera, quae etiam locos geometricos appellant, quae singula suam in se admodum simplicem, licet plerumque nobis compositam plurimum, et implexam, naturam habent, ...".

[10] Rogerius Josephus Boscovich, "De transformatione locorum geometricorum, ubi de continuitatis lege, ac de quibusdam Infiniti mysteriis," in Boscovich, *Elementorum universae matheseos,* Vol. 3 (Romae: Salomoni, 1754), pp. 297-468, nn. 673-886, on pp. 297-298, n. 674: "In primis quaecunque cujuscunque geometrici loci pars eandem naturam habet, quae ipsius definitione continetur;"

[11] Rogerius Josephus Boscovich, "Sectionum conicarum elementa," in Boscovich, *Elementorum universae matheseos,* Vol. 3 (Romae: Salomoni, 1754), pp. 1-296, nn. 1-672, on p. 1, n. 1, Def. I.

[12] Boscovich, "De transformatione locorum geometricorum" (cit. n. 10), n. 693, p. 313.

[13] Rogerius Josephus Boscovich, "Auctoris praefation," in Boscovich, *Elementorum universae matheseos,* Vol. 3 (Romae: Salomoni, 1754), pp. III-XXVI, on p. XVIII: "Ubi nimirum problema quodpiam generaliter solveris; mutata nonnihil datorum dispositione; plerumque ipsa constructio mutari plurimum debet, ...". Cfr. Željko Marković, *Rude Bošković,* Vol. 1 (cit. n. 1), p. 285; Žarko Dadić, *Ruđer Bošković* (Zagreb, Školska knjiga, 1987), pp. 195-196.

(3) the appearance of impossible or imaginary quantities;
(4) removing the point to infinity, whether it is the intersection of straight lines, the centre of a circle or, quite generally, the solution of a problem.

This elucidates what changes in the disposition of given quantities are concerned.

The treatise *De transformatione locorum geometricorum* can be thematically divided into two parts.[14] In the first part "a certain material prepared for a new building" *(materia quadam novi cujusdam aedificii praeparata)*,[15] namely for the theory of transformations, is presented. The second part, including nn. 759-886, contains the very theory in accordance with Bošković's views on how a geometrical theory should be constructed: first, the definition of the twofold analogy, and then eleven canons. The definition and the canons form the axiomatic system of Bošković's geometric theory. The material gathered for the theory and the very theory are structured. This can be seen from the introductory paragraphs of both parts of the treatise: from n. 673 where the presentation of the material begins, and from n. 759, which really represents the programmatic introduction to the general theory of the transformations of continuous curves.

Bošković's approach to the research into the transformations of geometric loci is constant throughout the treatise. While choosing the material and, also, while constructing his theory of geometric transformations, Bošković attempted to study in detail the law of geometrical continuity *(continuitatis geometricae lex)* and tried to explain some mysteries of infinity *(quaedam infiniti mysteria)* seeing in them a wonderful ability of geometry *(mira Geometriae indoles)*.[16] The same intention is obvious in some other places. The subtitle of the treatise is: *ubi de continuitatis lege, ac de quibusdam Infiniti mysteriis*. In the introductory paragraph of the treatise, as well as in the preface to the third volume of Bošković's *Elementorum universae matheseos*,[17] geometrical continuity and geometrical infinity are considered in their mutual relationship: the appearance of the mysteries of infinity brings up the demand that the continuity should be everywhere preserved and strictly observed, and vice versa, and where the behaviour of a curve is described by means of continuity, the mysteries of infinity still appear. Such a relationship between continuity and infinity is expected wherever the wonderful gift of geometry is mentioned, and this property is said to be manifest in every transformation of geometrical loci.

Bošković's clearly defined approach to the study of transformations can be followed through the basic thematic division of the treatise. The first part contains a survey of all qualitative forms of behaviour of continuous curves in geometric transformations, including the behaviour of continuous curves in infinity and in zero. Bošković simultaneously investigates the increase of a geometric object in infinity as a way of existence of the infinite and its vanishing as a way of existence of the infinitely small. This Bošković's

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[14] The first part of the treatise (cit. n. 10) includes nn. 673-758, pp. 297-367, and the second one includes nn. 759-886, pp. 367-468.

[15] Boscovich, "Auctoris praefatio" (cit. n. 13), p. XXI.

[16] Boscovich, "De transformatione locorum geometricorum" (cit. n. 10), n. 692, p. 312: "Interea earum [curvarum] ductus hìc definitus plurimum proderit ad quaedam infiniti mysteria evolvenda, et cognoscendam intimius continuitatis geometricae legem, ac ipsa plurimorum casuum contemplatio, et locorum generalis constructio sibi ubique respondens, ad Geometriae ipsius indolem, miram sanè, percipiendam pariter plurimum proderit."

[17] Boscovich, "De transformatione locorum geometricorum," n. 673, p. 297; Boscovich, "Auctoris praefatio," pp. XVIII-XX.

approach has been known since his early mathematical treatise *De natura et usu infinitorum et infinite parvorum,*[18] and here, it can be clearly seen, beginning from his study of a straight line as a continuos geometric creation.[19] The second part of the treatise *De transformatione locorum geometricorum* is already the realization of a new program of research into continuity and infinity in geometry.

The Axiomatic System of the Theory

The core of Bošković's research program is the axiomatic system which consists of the definition of the analogous geometric creation and of eleven canons for the transformations of geometric loci. Bošković is aware of the originality of his "new building" because he says in his treatise:

> It contains, however, a lot of things well worth knowing which I have not found anywhere else, and a lot of things that can also be found in other places but nowhere have I found them reduced to safe canons and studied by the geometric method.[20]

Still, he mentions that a similar system could be found in some unknown, very old papers.[21] This confirms once again that he founded his system of canons by means of the synthetic geometric method.

The construction of the axiomatic system of the theory Bošković commences with the definition of the primary and secondary analogy:

> 760. First of all, the points determined in the same way on both states of the same geometric construction, i.e. in the state before and after the transformation, are called the *analogous* points. They are, of course, determined by the intersection of the same geometric loci: straight lines by other straight lines, a circle, a perimeter of a conic section, lines defined by such an intersection according to the same law. ... However, we call *analogous* the lines terminated by two analogous points, the surfaces terminated by analogous lines, the solids terminated by analogous surfaces. ...
> 761. Then, we distinguish two kinds of this analogy. One is *primary* and complete, when, after the transformation, the direction of defined quantity remains or, it is changed in an even number of changes. The second kind of analogy is called *secondary* when the direction of quantity is changed suddenly or in an odd number of changes, and thus it can be called the *antianalogy*....[22]

In Bošković's supplementary explanation within the preface to the third volume of his *Elementorum universae matheseos,* this definition is called the definition of the duplex analogy *(duplicis analogiae definitio).*[23]

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[18] Rogerius Josephus Boscovich, *De natura et usus infinitorum et infinite parvorum* (Romae: Komarek, 1741), nn. 12-16, pp. 7-9.

[19] Boscovich, "De transformatione locorum geometricorum," n. 695, pp. 314-315.

[20] Boscovich, "Auctoris praefatio," p. XVIII: "Multa autem continet, quae licet scitu sane dignissima, ego quidem nusquam ego quidem ad certos reperi redacta canones, et geometrica methodo pertractata."

[21] Boscovich, "Auctoris praefation," l. c.: "Ea tamen pro novis venditare non audeo; cum mihi quidem inscitiae meae culpa, nova esse possint, licet fortasse sint apud Litterariam Remp [ublicam] vetustissima."

[22] Boscovich, "De transformatione locorum geometricorum," nn. 760-761, pp. 368-369, emphases in original.

[23] Boscovich, "Auctoris praefatio," p. XXI: "His expositis, et tanquam materia quadam novi cujusdam

This fundamental definition is followed by eleven canons. What are, generally speaking, Bošković's canons characterized by? It is clearly seen from the canons how Bošković differentiates, in every solved geometric problem, the following elements: proposition *(enuntiatio)*, proof *(demonstratio)*, and solution *(solutio)*. A canon shows what happens, in a transformation, to these three aspects of the geometric problem, i.e. what is changed in the proposition, proof and solution, if anything is changed at all. The first canon is a good example in this respect:

> 764. Canon I. If the quantities, upon which depend the solution of a problem or the proposition of a theorem, remain all analogous in terms of the first [primary] analogy after the transformation, and there is no transit through infinity, then the solution, proposition and proof remain the same, changed neither really nor formally. But, if we assume that some of them transit through infinity and are joined and connected in this same infinity, this finally leads to either of the following: in those quantities that depend only on direction everything remains the same; but in those pertinent to the magnitude it is necessary to estimate that ratio of theirs which derives from the law by which they are determined, and [which ratio] is wholly analogous to the one they would have if they do not transit through infinity.[24]

The canons are valid in the universal geometry.[25] The question arises how Bošković proves the canons while stressing their general validity? Bošković claims in the preface: "Some canons are proved exactly.",[26] and then he describes the character of the proof by terms as *exempla, applicatio, usus*. Are examples, application, and use the same as exact proof? Obviously, when constructing the theory of geometric transformations, Bošković's approach was different from the one he used when examining the fundamentals of the infinitesimal calculus at the beginning of his mathematical career.[27] In the treatise *De natura et usu infinitorum et infinite parvorum* (1741) he questioned the nature of basic calculus concepts and studied the counter-examples, such as the absurdity of the actual infinite for geometric extension, and the inflexion point of a cubic parabola, to such a degree that it became an epistemic barrier for the use of the infinitesimals in his mathematical investigations. This was why he did not systematically research into any field of their application. On the contrary, the nature of basic geometric quantities in Bošković's theory of geometric transformations is not questionable thanks to its Euclidean origin, and it is not questionable from the beginning of the construction of the theory. Therefore, there is a great possibility of their use in geometric transformations. The basic geometric quantities become questionable only in transformations, in fact, as a rule, in connection with infinity. A typical example, taken from Bošković's tenth canon, is a straight line understood as an infinite circle, and exactly this idea has a long Neoplatonic tradition.

According to Bošković's conception, whatever transformation of geometric locus can successfully be described by means of the transformations of basic geometric objects: quantities, proportions, and angles. Thus it is possible to classify the first nine canons of Bošković's theory into canons of quantities, canons of proportions, and canons

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aedificii praeparata, ad ordinandam transformationum theoriam progredior num. 760, quam duplicis analogiae definitione, et 11 Canonibus complector."

[24] Boscovich, "De transformatione locorum geometricorum," n. 764, p. 373.

[25] Boscovich, "De transformatione locorum geometricorum," n. 759, p. 368: "... canones, qui per universam late Geometriam observantur, ..."

[26] Boscovich, "Auctoris praefatio," p. XXIII: "Porro singuli Canones demonstrantur accurate."

[27] Cfr. Ivica Martinović, "Bošković's choice of method at the beginning of his mathematical career," *Dijalektika* 23 (1988), pp. 57-71.

of angles. The last two canons describe specific situations. The tenth canon considers the straight line as an infinite circle, and the eleventh discusses the comparison between the geometric infinites. Therewith two kinds of changes, which occur in geometric constructions as a result of transformations, are comprized in the theory: (1) appearance of impossible or imaginary quantities; (2) vanishing of the point in infinity, either as an intersection of straight lines or as a centre of a circle or, quite generally, as a solution of a geometric problem. Bošković thus attempts to construct such a system of canon which would completely describe all the changes arising from geometric transformations. The structure of the axiomatic system can be summarized in the following table:

Objects in transformation	Canons
(1) quantities	1, 2
(2) proportions	3, 6, 7
(3) angles	4, 5, 8, 9
(4) straight line as an infinite circle	10
(5) comparison between the geometric infinites	11

Bošković's Sources in the Construction of the General Theory of Transformations

The genesis of the theory of the transformations of geometric loci, as it was mentioned in the introduction, is closely connected with the transformations Bošković noticed while investigating conic sections. This is how Bošković described that relationship in his programmatic introduction into the theory:

> Hence, this very transformation of conic section is very suitable to declare and confirm some canons, which are widely observed in general geometry, and the examples of which should be taken from the proved elements of these curves. Indeed, from these canons, whose application to the same *Conicarum Sectionum Elementa* will be obvious, follows what is cammon to these curves, and what is supported by common proof, as well as what cannot be transfered from one [conic section] to another, and then the very reason of this anomaly is given. This will explain our intention to adorn these elements. In fact, such canons upon which, besides, as we have already noticed, constantly depend all [the elements of conic section], are like some fruit of theirs. We shall discuss them individually, reveal their meaning, and give the examples and the application to conic sections.[28]

According to Bošković's formulation, the theory of the transformations of geometric loci refers to the transformation of conic sections in two ways. First, the theory is considered to be *fruit* of proved transformations of conic sections. The relationship of the fundamentals of conic section towards the theory of transformations is the relationship of a constructed special theory towards a general one, which is yet to be constructed. The procedure carried out is *generalization,* and can be considered successful if we strictly observe the basic research aim in the construction of the theory, i.e. if some

[28] Boscovich, "De transformatione locorum geometricorum," n. 759, pp. 367-368.

behaviour of a continuous curve in infinity or in zero is declared in a general way. Then it takes the opposite direction. The theory of transformations *applies* to the fundamentals of conic section, and, if it is possible, to cases which were not an immediate inspiration while formulating the canons of the theory. The difficulties appearing in applications are especially studied. The procedure carried out in such a way is the procedure of confirmation or *verification* of the theory. This procedure was used by Bošković in his theory of forces as early as 1748, and it proved fruitful.[29]

To tell the truth, the transformations of conic sections are the fundamental, but not the only sources for the development of the general theory of transformations; they are the basic, but not the only field of application of this theory. Among additional sources we should certainly include the doctrine *coincidentia oppositorum* of Nicholaus of Cusa and the final scholium of Newton's method of prime and ultimate ratios.[30]

Conclusion

Induced by his study of conic section, Bošković constructed his own theory of the transformations of geometric loci in the period 1747-1753, and published the results of his research work in the treatise *De transformatione locorum geometricorum* (1754). Here, he used the model which was successful while he was developing his theory of forces. He conceived the forces in nature on the basis of the relationship between the fundamentals of mechanics and his own theory of forces, and in the same way he comprehended the transformation of a continuous curve, that is, on the basis of the relationship between the fundamentals of conic sections and his own theory of geometric transformations. Simultaneously, he wanted to widen the understanding of continuity and infinity in geometry. Moreover, by creating his system of canons, Bošković offered axiomatics for the systematic research on geometric transformations. Thus, what he put in the centre of the mathematical concern was not a geometric creation, but just the *transformation* of that geometric creation.

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[29] For a fuller discussion of the relationship between the foundations of mechanics and Bošković's theory of forces see Ivica Martinović, "The fundamental deductive chain of Bošković's natural philosophy," in Valentin Pozaić (ed.), *The philosophy of Ruđer Bošković* (Zagreb: Institute of Philosophy and Theology S. J., 1987), pp. 65-99, on pp. 84-93.

[30] See notes 7 and 8.

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